Draft version January 16, 2019
Typeset using LaTeX **RNAAS** style in AASTeX62

Hamiltonian Splittings with Jacobi and Democratic Heliocentric Coordinates

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Keywords: planets and satellites: dynamical evolution and stability — methods: numerical

Wisdom-Holman (WH) type integrators (Wisdom & Holman 1992) make the long term evolution of planetary systems computationally tractable by decomposing the motion into a dominant Keplerian part and a small perturbation part mainly originating from planet-planet interactions. We compare different splitting and coordinate system choices for WH integrators.

In Jacobi coordinates, the positions Q_i and momenta P_i of the ith particle are referenced to the center-of-mass of all interior particles with index $j < i$. We can *split* the N-body Hamiltonian as:

$$H = \underbrace{\frac{P_0^2}{2M}}_{H_0} + \underbrace{\sum_{i=1}^{N}\left(\frac{P_i^2 M_i}{2m_i M_{i-1}} - \frac{GM_{i-1}m_i}{Q_i}\right)}_{H_K} + \underbrace{\sum_{i=1}^{N}\frac{GM_{i-1}m_i}{Q_i} - \sum_{i=0}^{N}\sum_{j=i+1}^{N}\frac{Gm_im_j}{R_{ij}}}_{H_I}, \tag{1}$$

where $M = \sum_{j=0}^{N} m_j$, $M_i = \sum_{j=0}^{i} m_j$, and R_{ij} is the distance between particles i and j in standard Cartesian coordinates[1]. H_0 describes the motion of the barycentre and is trivial to solve. The i-th term in the Keplerian part H_K describes the evolution of the i-th planet around the centre of mass of the star and all interior planets. In the unperturbed limit where all masses other than those of the star and the first planet go to zero, the Keplerian motion for the first planet is governed entirely by H_K as $H_I = 0$. In an operator splitting scheme the 2-body problem is therefore solved exactly.

Alternatively, one can use the WH integrator with democratic heliocentric coordinates (WHD), where the positions $\tilde{Q}_i$ are relative to the star, and the corresponding canonical momenta $\tilde{P}_i$ are relative to the barycentre. Duncan et al. (1998) split the Hamiltonian as:

$$H = \underbrace{\frac{\tilde{P}_0^2}{2M}}_{H_0} + \underbrace{\sum_{i=1}^{N}\left(\frac{\tilde{P}_i^2}{2m_i} - \frac{Gm_0m_i}{\tilde{Q}_i}\right)}_{H_K} \underbrace{-\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{Gm_im_j}{\tilde{Q}_{ij}}}_{H_I} + \underbrace{\frac{1}{2m_0}\left(\sum_{i=1}^{N}\tilde{\mathbf{P}}_i\right)^2}_{H_J}. \tag{2}$$

The additional jump term H_J involves the momenta of the planets. In an operator splitting scheme such as WH, even a two body problem is not solved exactly anymore due to this term. This becomes significant for orbits with small periastron distances. An important advantage of this splitting is that the planets are treated equally and do not have to be ordered.

Hernandez & Dehnen (2017) propose WHDS, a slightly different splitting which also uses democratic heliocentric coordinates:

$$H = \underbrace{\frac{\tilde{P}_0^2}{2M}}_{H_0} + \underbrace{\sum_{i=1}^{N}\left(\frac{\tilde{P}_i^2(m_0+m_i)}{2m_im_0} - \frac{Gm_0m_i}{\tilde{Q}_i}\right)}_{H_K} \underbrace{-\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{Gm_im_j}{\tilde{Q}_{ij}}}_{H_I} + \underbrace{\frac{1}{2m_0}\left(\sum_{i=1}^{N}\sum_{j=i+1}^{N}\tilde{\mathbf{P}}_i\cdot\tilde{\mathbf{P}}_j\right)}_{H_J}. \tag{3}$$

[1] Slight variations in the precise choice of masses exist (Hernandez & Dehnen 2017) but this has no implications for the discussion here.

An additional advantage of WHDS is that it solves the Keplerian motion of all planets exactly in the limit of no perturbations.

There are several distinct scenarios one might be interested in. First, let us consider a planetary system where orbits never cross and always remain well separated. Jacobi coordinates are clearly the best choice here because all planets are effectively on Keplerian orbits around the centre of mass of all the interior bodies. If we were to use either WHD or WHDS we would have additional errors from the jump term.

For the second scenario, let us consider a system where close encounters between planets occur. Jacobi coordinates are not a good choice here because there is no physical way to order the planets anymore. As planets swap orbits due to close encounters, the gravitational masses appearing in H_K lose meaning. Although neither WHD nor WHDS can solve the Keplerian motion of all planets correctly at all times either, their democratic heliocentric coordinates have no preferred ordering and are thus better suited. An important advantage of WHD over WHDS is that the jump term changes the positions of all planets (but not the star) by the same amount, and thus the relative distances between planets only change during the Kepler step. When using WHDS on the other hand, the relative distances between planets also change during the jump step. Thus, during very close encounters between planets, the planets can *jump onto* each other during the jump step, leading to large errors because the interaction terms between planets are large.

The third scenario is a system where planets come very close to the star. Jacobi coordinates can only solve the Keplerian orbit around the star correctly for the innermost planet. The outer planets do not use the appropriate gravitational masses during the encounter. When using WHD, the jump step leads to large errors (because the planet can *jump onto* the star). Although WHDS can solve encounters with the star exactly in the limit of only one massive planet, if there are multiple massive planets, planets can still *jump onto* the star during the jump step.

Note that this discussion is also valid for hybrid symplectic integrators such as Mercury (Chambers 1999) where parts of H_I are being transferred to H_K during close encounters. The jump steps of WHD and WHDS remains unchanged in hybrid integrators and planets might still *jump onto* another body during it. This also complicates the detection of physical collisions because collisions can occur either during the Kepler or during the jump step. WHD has this issue only when dealing with planet-star collisions but not for planet-planet collisions, whereas WHDS has this issue for all collisions.

This leads us to conclude that in most reasonable scenarios (in particular well separated orbits and crossing orbits), either Jacobi coordinates or the standard WHD splitting have highly desirable properties. There is always a tradeoff when choosing a Hamiltonian splitting and no single splitting can solve every scenario well. If one's aim is to integrate every scenario well, then Wisdom-Holman type integrators are simply not the right tool and one is better off with a brute-force high order integrator (Rein & Spiegel 2015). Note that the discussion changes somewhat when considering massless test particles (Wisdom 2017).

This research has been supported by the NSERC Discovery Grant RGPIN-2014-04553 and the Centre for Planetary Sciences at the University of Toronto Scarborough. Support for this work was also provided by NASA through the NASA Hubble Fellowship grant HST-HF2-51423.001-A awarded by the Space Telescope Science Institute, which is operated by the Association of Universities for Research in Astronomy, Inc., for NASA, under contract NAS5-26555.

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